

**UNCLASSIFIED**

**AD** **409 943**

**DEFENSE DOCUMENTATION CENTER**

**FOR**

**SCIENTIFIC AND TECHNICAL INFORMATION**

**CAMERON STATION, ALEXANDRIA, VIRGINIA**



**UNCLASSIFIED**

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

CATALOGED BY DDC 409943

FTD-TT- 63-374

409 943

63-4-2

AS AD No.

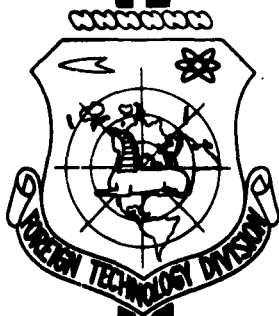
# TRANSLATION

EQUILIBRIUM OF A TOROIDAL PINCH IN A MAGNETIC FIELD

By

V. D. Shafranov

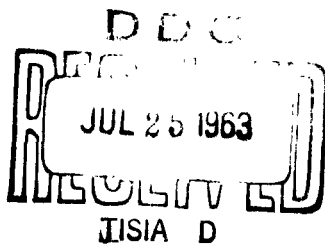
## FOREIGN TECHNOLOGY DIVISION



AIR FORCE SYSTEMS COMMAND

WRIGHT-PATTERSON AIR FORCE BASE

OHIO



## UNEDITED ROUGH DRAFT TRANSLATION

EQUILIBRIUM OF A TOROIDAL PINCH IN A MAGNETIC FIELD

By: V. D. Shafranov

English Pages: 20

Source: Russian Periodical: Atomnaya Energiya,  
Akademiya Nauk SSSR, Vol. 13, No. 6,  
1962, pp. 521-529.

T-84  
SOV/89-62-13-6-

THIS TRANSLATION IS A RENDITION OF THE ORIGINAL FOREIGN TEXT WITHOUT ANY ANALYTICAL OR EDITORIAL COMMENT. STATEMENTS OR THEORIES ADVOCATED OR IMPLIED ARE THOSE OF THE SOURCE AND DO NOT NECESSARILY REFLECT THE POSITION OR OPINION OF THE FOREIGN TECHNOLOGY DIVISION.

PREPARED BY:

TRANSLATION SERVICES BRANCH  
FOREIGN TECHNOLOGY DIVISION  
WPAFB, OHIO.

## EQUILIBRIUM OF A TOROIDAL PINCH IN A MAGNETIC FIELD

V. D. Shafranov

The conditions of equilibrium of an axisymmetric toroidal pinch are obtained by analysis with respect to  $a/R$  ( $a$  and  $R$  — the small and large radius of torus) without concrete representation of the type of current distribution and of longitudinal magnetic field with respect to the pinch profile. It is assumed only, that in a zero approximation ( $R = \infty$ ) the pinch has a cylindrical symmetry.

Refined formulas are given for pinch displacement in a conducting housing that are suitable in the second approximation, i.e., taking into account terms of the order  $b^2/R^2$ , where  $b$  is the housing radius.

### Introduction

The equilibrium conditions of a toroidal axisymmetric pinch in a magnetic field were derived in earlier articles [1-6] under certain hypothesis of the current distribution with respect to a pinch cross section. In previous articles [1-5] idealized configurations either with a surface current or with a uniform current were examined. One article [6] cited the solution for the case where the surface current distribution with respect to the cross section was described as a zero Bessel function (from an actual or imaginary argument). Below,

the equilibrium conditions are obtained without concrete representation of the type of current density distribution or longitudinal magnetic field with respect to the section. Presented only is that  $a \ll R$  and that in the zero approximation ( $R = \infty$ ) the distribution is cylindrically symmetrical.

To find the magnetic field confining the pinch in equilibrium, we must know the field distribution on the pinch surface. Thus, for example, in an idealized case of a surface current and with no longitudinal magnetic field, the pressure of the plasma in the pinch is constant  $p = \text{const}$ , and consequently the meridional magnetic field at the pinch surface is constant. This condition permits us to find the general magnetic field outside of the plasma [2]. In the case of space currents, the meridional magnetic field at the pinch surface is already variable, but when  $\frac{a}{R} \ll 1$  it can, apparently, be expressed in terms of such characteristics as internal self-induction of a unit of pinch length  $l_1$  and average pressure  $\bar{p}$  of the plasma with respect to the section. For the surface of the pinch with distributed current we can assume an arbitrary magnetic surface on which the current density is negligible as compared with the average current density in the pinch. In the zero approximation ( $R = \infty$ ) magnetic surfaces are a system of concentric cylinders ( $\rho = \text{const}$ ). In the first approximation with respect to  $a/R$  the magnetic surfaces are a system of enclosed toruses whose cross sections remain as circles but are no longer concentric.

In this article the magnetic field distribution over magnetic surface is derived in part 2 by examining the balance of forces acting on a volume element of wedge-shaped form, formed by the rotation of a sector of the section of magnetic surface around the axis of

symmetry of the torus. The boundary condition (21) derived in that section is used in part 3 for calculation of the external magnetic field needed to confine an annular pinch in equilibrium. In part 4, using the same condition (21), we obtain the formulas describing the equilibrium of a toroidal pinch inside a conducting housing. The formulas for magnetic field distribution outside the plasma and for the displacement value of the center of the plasma section relative to a section of the housing are presented. We will examine the distortion of the shape of its pinch section associated with the toroidal effect and the influence of various factors (extraneous factors, "scattered" fields, separations in the housing, the finite electrical conductivity of the latter) on the equilibrium of the pinch.

## 2. Magnetic Field Distribution on a Magnetic Surface

The equation of equilibrium of plasma in a magnetic field

$$-\nabla p + \frac{1}{c} [\mathbf{jB}] = 0 \quad (1)$$

can be written, as is known, by means of the tensor of maxwellian stresses in an integral form

$$\oint T_{\alpha\beta} dS_\beta = 0, \quad (2)$$

where

$$T_{\alpha\beta} = \left( p + \frac{B^2}{8\pi} \right) \delta_{\alpha\beta} - \frac{B_\alpha B_\beta}{4\pi}. \quad (3)$$

Here integration in (2) is performed with respect to the surface arbitrarily bounding an isolated volume. To determine the magnetic field on a magnetic surface, whose section radius equals a, we apply relation (2) to the volume element which is bounded by coordinate

STOP HERE

STOP HERE

surfaces (Fig. 1).

$$\varphi + d\varphi = \text{const}; 0 < \varphi < a. \quad (4)$$

The coordinates  $\rho$ ,  $\varphi$ ,  $\omega$  are connected with the cylindrical coordinates  $r$ ,  $\varphi$ ,  $z$  by the relation

$$r = R + \varrho \cos \omega; \quad z = \varrho \sin \omega, \quad (5)$$

where  $R$  is the distance of the center of the section of the examined magnetic surface from the axis of symmetry. The corresponding unit vectors are determined by the relations

$$\left. \begin{aligned} e_\varphi &= e_r \cos \omega + e_z \sin \omega, \\ e_\omega &= e_z \cos \omega - e_r \sin \omega, \\ \frac{\partial e_\omega}{\partial \omega} &= -e_\varphi, \quad \frac{\partial e_\varphi}{\partial \omega} = e_\omega. \end{aligned} \right\} \quad (6)$$

We will carry out the calculation in the first approximation of expansion with respect to  $a/R$ . In the zero approximation ( $R = \infty$ ) we will deal with a cylindrical pinch, for which the relation [7]

$$\bar{p} + \frac{\overline{B_1^2}}{8\pi} = \frac{B_e^2}{8\pi} + \frac{B_c^2}{8\pi} + p_a. \quad (7)$$

is valid.

Here  $B_1$  and  $B_e$  are the internal and external longitudinal fields;  $B_a = \frac{2J}{ca}$  is the current field;  $p_a$  is the pressure where  $\rho = a$ . The line indicates averaging over the pinch section  $0 < \rho < a$ . In the first approximation terms of the type  $B^0 B^{(1)}$  appear in tensor  $T_{\alpha\beta}$ , wherein  $B^{(1)} \sim \frac{\cos \omega}{\sin \omega}$ . Since on averaging over a section of the toroidal pinch such terms disappear, then (7) holds true also in the first approximation if  $B_e$  is taken as the value of the longitudinal magnetic field on a magnetic surface with radius  $a$  for  $\omega = \frac{\pi}{2}$ . As is known, in an axisymmetrical pinch the longitudinal magnetic field changes along

the magnetic surface  $\psi$ ,  $\rho$ ,  $\omega = \text{const}$  with respect to the law [1]

$$B_\varphi = \frac{2I(\psi)}{cr} = \frac{2I(\psi)}{cR} \left(1 - \frac{c}{R} \cos \omega + \dots\right) = B_0(\varphi) \left(1 - \frac{c}{R} \cos \omega + \dots\right). \quad (8)$$

The pressure of plasma on the magnetic surface maintains a constant value  $p = p(\psi)$ .

We will designate by  $dS_\rho$ ,  $dS_\varphi$ ,  $dS_\omega$  the surface elements the normals to which are unit vectors  $e_\rho$ ,  $e_\varphi$ ,  $e_\omega$ . The resultant of forces acting on the surfaces  $dS_\varphi(\varphi)$  and  $dS_\varphi(\varphi + d\varphi) = dS_\varphi(\varphi) = \rho d\rho d\omega$ , is

$$F_{\varphi\alpha} = -d\varphi d\omega \int_0^a T_{\varphi\alpha} \frac{\partial e_\alpha}{\partial \varphi} \varrho d\varrho, \quad (9)$$

where summation with respect to  $\alpha = \rho, \varphi, \omega$  is inferred. This is a force of the first order of smallness, therefore tensor  $T_{\varphi\alpha}$  should be calculated only in the zero approximation:

$$F_{\varphi\alpha} = F_{\varphi\alpha}^{(0)} = -d\varphi d\omega \int_0^a [T_{\varphi\varphi}^0 (e_\omega \sin \omega - e_\varphi \cos \omega) - T_{\varphi\omega}^0 c_\varphi \sin \omega] \varrho d\varrho. \quad (10)$$

The resultant of forces acting on surfaces  $dS_\omega(\omega)$  and  $dS_\omega(\omega + d\omega)$  is analogously determined. Here we need only calculate that on changing from  $\omega$  to  $\omega + d\omega$  both the unit vectors and  $T_{\omega\alpha}$  and also  $dS_\omega = (R + \rho \cos \omega) d\rho d\varphi$  change. We will present the result:

$$F_{\omega\alpha} = F_{\omega\alpha}^0 + F_{\omega\alpha}^{(1)} = d\varphi d\omega \left( R e_\varphi \int_0^a T_{\omega\omega}^0 d\varrho \right) + d\varphi d\omega \left\{ e_\varphi \left[ \cos \omega \int_0^a T_{\omega\omega}^0 d\varrho + R \int_0^a \left( T_{\omega\omega}^{(1)} - \frac{\partial T_{\omega\omega}^{(1)}}{\partial \omega} \right) d\varrho \right] + e_\omega \left[ \sin \omega \int_0^a T_{\omega\omega}^0 d\varrho - R \int_0^a \left( T_{\omega\varphi}^{(1)} + \frac{\partial T_{\omega\varphi}^{(1)}}{\partial \omega} \right) d\varrho \right] + e_\varphi \left[ \sin \omega \int_0^a T_{\omega\varphi}^0 d\varrho - R \int_0^a \frac{\partial T_{\omega\varphi}^{(1)}}{\partial \omega} d\varrho \right] \right\}. \quad (11)$$

Finally, taking into account that  $B_p(a) = 0$ , and consequently  $T_{p\omega}(a) = T_{p\varphi}(a) = 0$ , we will find

$$\begin{aligned} F_{\omega} &= -e_0 T_{\omega\omega} (R + a \cos \omega) a d\varphi d\omega = \\ &= e_0 [-T_{\omega\omega}^* R a + (T_{\omega\omega}^{(1)} R a + \\ &+ T_{\omega\omega}^* a^2 \cos \omega)] \equiv F_{\omega}^* + F_{\omega}^{(1)}. \end{aligned} \quad (12)$$

Setting the sum of these three forces (10)-(12) equal to zero, we will obtain four conditions of equilibrium of an isolated volume element: equilibrium with respect to  $p$  in the zero approximation and equilibrium with respect to  $\varphi$ ,  $\omega$  and  $p$  in the first approximation:

$$\int T_{\omega\omega}^* dQ = T_{\omega\omega}^*(a) a; \quad (13)$$

$$\frac{\partial}{\partial \omega} \int T_{\omega\varphi}^{(1)} dQ = \frac{2 \sin \omega}{R} \int T_{\omega\varphi}^* Q dQ; \quad (14)$$

$$\int \left( T_{\omega\omega}^{(1)} + \frac{\partial T_{\omega\omega}^{(1)}}{\partial \omega} \right) dQ = \frac{\sin \omega}{R} \int (T_{\omega\omega}^* - T_{\varphi\varphi}^*) Q dQ; \quad (15)$$

$$\begin{aligned} \cos \omega \int (T_{\varphi\varphi}^* + T_{\omega\omega}^*) Q dQ - R \int \left( \frac{\partial T_{\omega\omega}^{(1)}}{\partial \omega} - T_{\omega\omega}^{(1)} \right) \times \\ \times dQ - T_{\omega\omega}^* a^2 \cos \omega - T_{\omega\omega}^{(1)} R a = 0. \end{aligned} \quad (16)$$

The equilibrium condition in the zero approximation (13) can be obtained after having averaged (1) over the plasma section. Conditions (14), (15) can be written differently if we consider that  $\frac{\partial^2 T^{(1)}}{\partial \omega^2} = -T^{(1)}$ , since  $T^{(1)} \sim \frac{\sin \omega}{\cos \omega}$ . Thus, for example, (15) can be written as

$$\int \left( \frac{\partial T_{\omega\omega}^{(1)}}{\partial \omega} - T_{\omega\omega}^{(1)} \right) dQ = \frac{\cos \omega}{R} \int (T_{\omega\omega}^* - T_{\varphi\varphi}^*) Q dQ. \quad (17)$$

Eliminating the second integral in (16) with the help of (17) we will obtain

$$\begin{aligned} T_{\omega\omega}^{(1)}(a) &= \frac{a}{R} \cos \omega \left[ -T_{\omega\omega}^*(a) + \right. \\ &\left. + \frac{2}{a^2} \int T_{\varphi\varphi}^* Q dQ \right]. \end{aligned} \quad (18)$$

In the examined approximation

$$T_{\omega\omega} = p + \frac{B_{\omega}^2}{8\pi} + \frac{B_{\varphi}^2}{8\pi}; \quad T_{\varphi\varphi} = p + \frac{B_{\omega}^2}{8\pi} - \frac{B_{\varphi}^2}{8\pi} \quad (19)$$

(the term  $B_{\varphi}^2/8\pi$  is dropped since it is of the second order of smallness). Using (8), we will find

$$T_{\omega\omega}^{(1)}(a) = \frac{B_a B_{\omega}^{(1)}}{4\pi} - \frac{B_a^2}{8\pi} 2 \frac{a}{R} \cos \omega. \quad (20)$$

From (18)-(20) and (7) we will obtain the unknown meridional field on the magnetic surface with radius  $a$ :  $B_{\omega}(a) = B_{\omega} + B_{\omega}^{(1)}$ , where

$$B_{\omega}^{(1)} = \frac{a}{R} B_a \cos \omega \left[ \frac{l_1}{2} + \frac{8\pi (\bar{p} - p_a)}{B_a^2} - 1 \right]. \quad (21)$$

Here the internal self-induction of a unit plasma length is

$$l_1 = \frac{1}{\pi a^2 B_a^2} \int_0^{\pi} B_{\omega}^2 d\varphi d\omega \quad (22)$$

(for surface current, for example, it is  $l_1 = 0$ ; for uniform current it is  $l_1 = 1/2$ ).

We note that for the derivation of (18) a concrete representation of tensor  $T_{\alpha\beta}$  was not used; only the condition  $T_{\rho\omega}(a) = T_{\rho\varphi}(a) = 0$ , was used, therefore, Expression (18) is applicable not only to hydromagnetic equilibrium configurations. If, for example, we take  $T_{\alpha\beta} = p\delta_{\alpha\beta} + \rho v_{\alpha} b_{\beta}$ , as the tensor  $T_{\alpha\beta}$ , then Condition (18) permits us to find a hydrodynamic flow originating about an annular vortex.

Relation (21) is correct for any magnetic surface in an equilibrium configuration, whose section radius is  $a \ll R$ . Subsequently this condition will be applied to a "pinch surface," i.e., applied to that magnetic surface outside of which the magnetic current density can be taken as equal to zero.

### 3. Equilibrium Without External Conductors

A meridional magnetic field outside of the pinch can be written as

$$B = \frac{1}{2\pi r} [\nabla \psi e_{\phi}], \quad (23)$$

wherein the lines of magnetic force are arranged on the surface  $\psi = \text{const.}$  When  $a \ll R$  [4]

$$\psi = [2(\cosh \vartheta - \cos \tilde{\omega})]^{-1/2} \times [F_0(\vartheta) + 2F_1(\vartheta) \cos \tilde{\omega}], \quad (24)$$

where

$$\begin{aligned} F_0(\vartheta) &= a_0 g_0(\vartheta) + b_0 f_0(\vartheta); \\ F_1(\vartheta) &= a_1 g_1(\vartheta) + b_1 f_1(\vartheta). \end{aligned}$$

Here  $a_0, b_0, a_1, b_1$  are the constants subject to determination;  $g_0, g_1, f_0, f_1$  are the Fok toroidal functions [8];  $\vartheta, \omega$  are the toroidal coordinates which when  $\rho \ll R$  are connected with  $\rho$  and  $\omega$  by the relations

$$\tilde{\omega} = \omega, \quad e^{-\vartheta} = \frac{\rho}{2R} + \frac{a^2}{4R^2} \left(1 - \frac{\rho^2}{a^2}\right) \cos \omega. \quad (25)$$

When  $\rho \ll R$

$$\begin{aligned} \psi &= a_0 + \frac{2}{\pi} b_0 (\vartheta + 2 \ln 2 - 2) + \\ &+ \left\{ \left[ a_0 + \frac{2}{\pi} b_0 (\vartheta + 2 \ln 2 - 2) \right] e^{-\vartheta} + \right. \\ &\left. + \left( \frac{4}{3\pi} b_1 e^{\vartheta} - a_1 e^{-\vartheta} \right) \right\} \cos \omega. \end{aligned} \quad (26)$$

The confining field  $\psi^{re}$  is obtained when  $b_0 = b_1 = 0$ :

$$\begin{aligned} \psi^{re} &= a_0 + (a_0 - a_1) e^{-\vartheta} \cos \omega = \\ &= a_0 + \frac{a_0 - a_1}{2R} (r - R). \end{aligned} \quad (27)$$

The corresponding magnetic field is obtained by the formulas

$$B_r^{re} = 0; \quad B_z^{re} = \frac{1}{2\pi r} \frac{\partial \psi^{re}}{\partial r} = \frac{a_0 - a_1}{4\pi R^2}. \quad (28)$$

The constants  $a_0, b_0, a_1, b_1$  determined from the conditions of constancy  $\psi$  and continuity  $B_\omega$  on the pinch surface, are (assuming  $\psi_p = a = 0$ ) equal to

$$\begin{aligned} a_0 &= 2\pi R a B_a \left( \ln \frac{8R}{a} - 2 \right); \quad b_0 = -\pi^2 R a B_a; \\ a_1 &= -2\pi R a B_a \left( \frac{1+l_1}{2} + \frac{8\pi \bar{p}}{B_a^2} \right); \\ b_1 &= \frac{3\pi}{16} \frac{a^2}{R^2} a_1. \end{aligned} \quad (29)$$

Thus, in agreement with Expressions (28) and (29), the external magnetic field needed to confine the plasma in equilibrium is

$$B_z^{re} = \frac{a}{2R} B_a \left[ \ln \frac{8R}{a} - \frac{3}{2} + \frac{l_1}{2} + \frac{8\pi (\bar{p} - p_0)}{B_a^2} \right]. \quad (30)$$

In particular, from this formula the values of  $B_z^{re}$  found in earlier articles [1-6] are obtained. Assuming  $B_z^{re} = 0$ , from (30) and (7) we will find the conditions of equilibrium of a magnetic field in a toroidal space in the case where the magnetic field immersed in plasma is analogous to the magnetic field in sun spots

$$8\pi p_0 = 8\pi \bar{p} + B_a^2 \left( \ln \frac{8R}{a} - \frac{3}{2} - \frac{l_1}{2} \right); \quad (31)$$

$$\bar{B}_1^2 = B_a^2 \left( \ln \frac{8R}{a} - \frac{1}{2} + \frac{l_1}{2} \right).$$

In a limiting case of surface current ( $l_1 = 0, p = \text{const}$ ) these formulas agree with the results obtained in a previous article [1].

#### 4. Equilibrium Inside a Conducting Housing

Important in practice is the situation of equilibrium in a conducting housing which is used, for example, in an experimental "Tokamak"

system intended to create and study high-temperature plasma. The forces resulting from the removal of the line current from the housing center was calculated by M. A. Lentovich [9, 10]. In one of the articles [9] cases are examined where the thickness of skin layer is significantly less and significantly greater than the housing wall thickness. Another article [10] took into account influence of the housing joints needed to lead the electrical field into the discharge chamber. These articles [9, 10] solve the principal problem of the possibility of confining a pinch inside a toroidal conducting housing. By using the expression for forces obtained in these articles [9, 10], we can estimate the displacement value  $\Delta$  of the pinch inside the housing [11]. The experiments on "Tokamak" showed that certain substantial peculiarities of discharge are apparently associated with a shift of the pinch during discharge which leads to contact of the plasma with the diaphragm confining the discharge [12]. Thus the problem arises of finding a more precise value of the pinch displacement in the housing. Expression (21), which was found above for the meridional field on the pinch surface, yields the solution to this problem. Below, expressions are obtained for pinch displacement in a conducting housing in the second approximation of expansion with respect to  $b/R$  and also determined is the distortion of the shape of the pinch section caused by additional so-called scattered fields.\*

In the second approximation of expansion with respect to  $1/R$ , the pinch section will no longer be round but will have an elliptical form

$$\varrho = a + \delta \cos 2\alpha. \quad (32)$$

---

\* L. A. Artsimovich acquainted the author with the last problem.

We will assume that the housing section in the general case also has an elliptical form. In a polar system of coordinates with an origin in the center of the housing section, let the equation of the surface of the latter be

$$\rho' = b + \kappa \cos 2\omega'. \quad (33)$$

Then in the system of coordinates  $\rho, \omega$  related to the pinch, whose cross section center is shifted from the center of the housing section (outward) to a distance  $\Delta$ , the equation of the housing surface has the form

$$\rho = b - \Delta \cos \omega - \frac{\Delta^2}{4b} + \left( \kappa + \frac{\Delta^2}{4b} \right) \cos 2\omega + \dots \quad (34)$$

The normals to the pinch and housing surfaces in the same approximation have the components

$$n_\rho = 1, \quad n_\omega = \frac{2\delta}{a} \sin 2\omega, \quad n_\varphi = 0; \quad (35)$$

$$\begin{aligned} n_\rho &= 1 - \frac{\Delta^2}{4b^2} (1 - \cos 2\omega), \quad n_\omega = \\ &= -\frac{\Delta}{b} \sin \omega + 2 \frac{\kappa}{b} \sin 2\omega, \quad n_\varphi = 0. \end{aligned} \quad (36)$$

respectively.

With the fulfillment of the conditions of ideal conductivity of the housing

$$\frac{c^2 t}{4\pi \sigma_K} \ll d^2 \quad (37)$$

( $t$  is the discharge time,  $\sigma_K$  and  $d$  are the conductivity and thickness of the housing wall) the normal component of the magnetic field, created by currents in the plasma, can be taken as equal to zero on the housing. From this condition,  $\Delta$  and  $\delta$  are determined. The influence of an additional magnetic field in the discharge chamber

can be calculated in the same manner. In this approximation and provided the symmetry is maintained, the additional field can be written as

$$\begin{aligned} B = \nabla \tilde{\Phi}, \quad \tilde{\Phi} = B_0 q \sin \omega + \\ + B_1 \frac{q^2}{2b} \sin 2\omega = B_0 z + B_1 z \frac{r-R}{b}. \end{aligned} \quad (38)$$

where  $B_0$  is the value of the magnetic field component normal to the housing surface in the center of its section:  $B_1$  characterizes the "barrel-shape quality" of the additional field. Disregarding whether the field extends to the housing or only to the conductors situated between the plasma and the housing, an effective boundary condition will be the equality between the normal component of field  $B$  outside the plasma, calculated without such conductors, and the value of the normal component of the additional field (38) extended to the housing i.e.

$$(B_n) = B_0 \sin \omega + \left( B_1 - \frac{\Delta}{2b} B_0 \right) \sin 2\omega. \quad (39)$$

The magnetic field outside of the plasma is determined from the equations

$$\left. \begin{aligned} B = \nabla \Phi; \\ \frac{\partial^2 \Phi}{\partial q^2} + \frac{1}{q} \frac{\partial \Phi}{\partial q} + \frac{1}{q^2} \frac{\partial^2 \Phi}{\partial \omega^2} = \\ = \frac{\sin \omega}{r} \frac{\partial \Phi}{\partial \omega} - \frac{\cos \omega}{r} \frac{\partial \Phi}{\partial q}. \end{aligned} \right\} \quad (40)$$

On the pinch surface by definition

$$(B_n) = 0. \quad (41)$$

The value of the  $\omega$ -component of the magnetic field on the pinch surface we will write as

$$B_\omega = B_0 \left( 1 + \frac{a}{R} \Lambda_1 \cos \omega + \frac{a^2}{R^2} \Lambda_2 \cos 2\omega \right). \quad (42)$$

In agreement with (21), the value is

$$\Lambda_1 = \frac{8\pi p}{B_0^2} + \frac{I_1}{2} - 1. \quad (43)$$

The precise value of  $\Lambda_2$  for further examination is immaterial.

The calculation of the magnetic field  $B$ , displacement  $\Delta$ , and distortion  $\delta$  are elementary; thus, below, we will introduce only the calculation result.

1. Magnetic Field Outside Plasma. In the examined approximation the solution of the equations in (40), on fulfilling boundary conditions in (41) and (42), has on the pinch surface the following form

$$B_\theta = \frac{\partial \Phi}{\partial \rho}, \quad B_z = \frac{\partial \Phi}{\partial z}, \quad (44)$$

where

$$\Phi = \Phi_0 \omega + \Phi_1 \sin \omega + \Phi_2 \sin 2\omega.$$

Here

$$\Phi_0 = B_0 a; \quad (45)$$

$$\Phi_1 = \frac{a}{2R} B_0 a \left[ \left( \Lambda_1 + \frac{1}{2} \right) \left( \frac{a}{\rho} + \frac{\rho}{a} \right) + \frac{\rho}{a} \left( \ln \frac{\rho}{a} - 1 \right) \right], \quad (46)$$

$$\begin{aligned} \Phi_2 = & \frac{a^3}{4R^2} B_0 a \left[ \frac{\delta}{a} \frac{R^2}{a^2} \left( \frac{3a^2}{\rho^2} - \frac{\rho^2}{a^2} \right) - \right. \\ & - \frac{1}{2} \left( \Lambda_1 + \frac{1}{2} \right) - \frac{3}{4} \frac{\rho^2}{a^2} \ln \frac{\rho}{a} + \\ & + \frac{a^2}{4\rho^2} \left( \Lambda_1 - \frac{1}{4} + 4\Lambda_2 \right) + \\ & \left. + \frac{\rho^2}{4a^2} \left( \Lambda_1 + \frac{5}{4} + 4\Lambda_2 \right) \right]. \quad (47) \end{aligned}$$

## 2. Displacement of a Pinch in a Ideally Conducting Housing.

The displacement  $\Delta$  in the second approximation (with respect to terms of order  $b^2/R^2$ ) agrees with the displacement calculated in the

second approximation and is determined by the formula\*

$$\Delta = \frac{b}{2R} \left[ \ln \frac{b}{a} + \left(1 - \frac{a^2}{b^2}\right) \left(\Lambda_1 + \frac{1}{2}\right) \right] - \frac{B_0}{B_0} \quad (48)$$

Here  $B_b = B_a \frac{a}{b} = \frac{2J}{cb}$ ;  $\Lambda_1$  is determined with respect to (43). The relation  $B_0/B_b$  agreeing with the usual reference frame is positive when the direction of the additional magnetic field  $B_0$  agrees with the direction of the field comprised of a current on the outside of the pinch.

In the "Tokamak" device, between plasma and housing there is a weak conducting chamber made of thin rust-proof steel through which flows the current equalizing the electrical field along the length of the discharge chamber. Let  $b_1$  be the radius of the chamber,  $\Delta_1$  be the displacement of the center of the chamber cross section relative to the center of the housing section in a direction from the axis of symmetry. Further let the resistance of the chamber in the section change with respect to the law  $1 + \lambda \cos \omega$  (for a bellows chamber  $\lambda = 0$ , for smooth chamber  $\lambda \simeq \frac{b_1}{R}$ ). Then the additional magnetic field created in the discharge chamber by current  $J_1$  flowing along the chamber, is

$$B_0 = -\frac{J_1}{cR} \left[ \ln \frac{b}{b_1} + \left( \frac{1}{2} - \frac{R\lambda}{b_1} \right) \left( 1 - \frac{b_1^2}{b^2} \right) - \frac{2R\Delta_1}{b^2} \right] \quad (49)$$

For the parameters of the "Tokamak-2" device [14] ( $2R = 125$ ,  $b_1 = 20$ ,  $b = 25$ ) the total displacement of the pinch (in centimeters) with a

---

\* For the idealized case of surface current and when  $B_0 = 0$  the corresponding calculation of displacement in the first approximation was done by V. M. Novikov in a thesis (1958) completed under the supervision of this article's author. The expression for  $\Delta$  where  $a \ll b$  and  $\Lambda_1 = 0$  is introduced in another article [11].

radius equal to the radius of the diaphragm ( $a = 10$ ), determinable (disregarding joints) by the formulas in (48), (49), is

$$\Delta = 2,5 + 4,2 \left( \frac{8\pi \bar{p}}{B_0^2} + \frac{l_1}{2} \right) + 1,9 \frac{J_1}{J}. \quad (50)$$

The additional displacement associated with the currents in the chamber can be eliminated by selecting appropriate values of  $\lambda$  and  $\Delta_1$ . When  $\Delta_1 = 0$  and  $b - b_1 \ll b_1$  it is necessary to have  $\lambda = \frac{b_1}{R}$  (this condition is fulfilled when the chamber is smooth). For the bellows chamber the additional displacement can be reduced to zero after having shifted the chamber to the outside housing wall to a distance of  $\Delta_1 = \frac{b}{R} (b - b_1)$ .\*

More hazardous is the shift associated with a pressure increase while heating the plasma; this shift with the parameters of the "Tokamak-2" reaches, as is obvious from (50), several centimeters when the pressure changes from zero to  $\bar{p} \simeq \frac{m^2}{8\pi}$ .\*\*

3. Distorted Shapes of a Pinch Section. The distortion amplitude  $\delta$  is determined by the relation

$$\begin{aligned} \frac{\delta}{a} = & -\frac{a^2}{2R^2} \left( 1 + \frac{3a^4}{b^4} \right)^{-1} \left[ \left( \ln \frac{b}{a} \right)^2 + \right. \\ & -2 \left( \Lambda_1 + \frac{5}{4} \right) \ln \frac{b}{a} + \left( 1 - \frac{a^4}{b^4} \right) \left( \Lambda_1^2 + \frac{\Lambda_1}{2} + \right. \\ & \left. \left. + \frac{3}{8} - 2\Lambda_2 \right) - \frac{8R^2}{b^2} \alpha \right] + \frac{\bar{\delta}}{a}. \end{aligned} \quad (51)$$

---

\* These conditions were obtained by A. I. Morozov and L. I. Solov'ev.

\*\* We notice that with a small plasma density the equilibrium conditions can be influenced by the centrifugal force of the current. for its calculation, we add to  $\Lambda_1$  the term  $\delta\Lambda_1 = \frac{mc^2}{e^2 N}$ , where  $N = \frac{J^2}{\int c^2 n v_{\phi}^2 dQ d\omega}$  (when  $v_{\phi} = \text{const}$ , the value  $N$  agrees with the linear number of discharges in the pinch).

where  $\tilde{\delta}$  is the distortion, caused by the supplementary scattered fields

$$\frac{\tilde{\delta}}{a} = \left(1 + \frac{3a^4}{b^4}\right)^{-1} \left[ \frac{a}{H} \frac{B_0}{H_0} \left( \ln \frac{b}{a} + \frac{1}{2} \right) - \frac{2B_0^2}{H_0^2} - \frac{2a}{b} \frac{B_1}{H_0} \right]. \quad (52)$$

The negative value of  $\delta$  corresponds to elongation of the pinch section along the axis of symmetry of the torus. When  $B_1 = 0$  the expression for  $\delta$  can be approximately written as

$$\frac{\delta}{a} \simeq -\frac{2a^2}{b^2} \frac{B_0^2}{H_0^2}. \quad (53)$$

Thus it is obvious that with a limited displacement value of ( $\Delta < b - a$ ) the comparative distortion is always very small ( $\delta/a \lesssim 1/8$ ). A noticeable distortion of the section is possible with a sufficiently large constituent of the quadrupole component of the additional field ( $B_1 \sim B_0$ ). The influence of this field when  $B_0 = 0$  is illustrated in Fig. 2. For the case shown in Fig. 2 ( $B_1 > 0$ ), this field is added to the current field on the lateral sides of the pinch and is subtracted from it on the upper and lower parts of the pinch.

With this orientation of the fields the pinch will be stretched out along the axis of symmetry. Outside of the plasma the general magnetic field change associated with distortion of the section is determined from Formulas (44)-(47) and has the form ( $\omega$ -component)

$$\delta B_\omega = B_0 \frac{\delta}{a} \left( \frac{3}{2} \frac{a^2}{b^2} - \frac{1}{2} \right) \cos 2\omega. \quad (54)$$

4. Effect of Joints on Pinch Displacement. When the housing has a joint the normal component of the magnetic field will be the function of  $\varphi$ . In the first approximation

$$(B_n) = f(\varphi) \sin \omega. \quad (55)$$

We will expand  $f(\varphi)$  in a Fourier series

$$f(\varphi) = B_0 + 2 \sum_{m=1}^{\infty} B_m \cos m\varphi. \quad (56)$$

The harmonics with  $m \geq 1$  lead to the appearance of heterogeneity (which we will not calculate here) of the pinch along its length, the zero harmonic

$$B_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) d\varphi \quad (57)$$

leads to a certain displacement of the column as a whole. For a zero harmonic, (55) has the form

$$(B_n) = B_0 \sin \omega, \quad (55a)$$

i.e., it agrees with the condition on the housing (39) in the presence of the additional field  $B_0$ . The value  $f(\varphi)$  (in the absence of some kind of screen) was determined with logarithmic accuracy in a previous article [10]. In our notations it is written as:

$$f(\varphi) = \begin{cases} -\frac{\Phi_1(b)}{2R \sqrt{\varphi_1^2 - \varphi^2} \ln \frac{ab}{h}}, & (-\varphi_1 < \varphi < \varphi_1) \\ 0, & (\varphi^2 > \varphi_1^2). \end{cases} \quad (58)$$

The width of the joint is taken as equal to  $2h$ ;  $\alpha$  is a coefficient of the order of unity [10];  $\varphi_1 = \frac{h}{R}$ ;  $\Phi_1(\rho) \sin \omega$  is the correction for the magnetic "potential" in the first approximation with an uninterrupted casing, i.e., in the examined case  $\Phi_1(\rho)$  is determined by Formula (46). From (57), (58) we will find

$$B_0 = -\frac{\Phi_1(b)}{2\pi R} \mu. \quad (59)$$

Here  $\mu = \frac{\pi}{2 \ln \frac{ab}{a}}$ . In the case of N joints separated from one another by a distance larger than  $b$ , we can assume

$$\mu = \sum_{i=1}^N \frac{\pi}{2 \ln \frac{a_i b}{a_i}}. \quad (60)$$

Substitution of (59) into the displacement formula in (48) gives

$$\begin{aligned} \frac{\Delta}{b} = \frac{b}{2R} \left\{ \left[ \ln \frac{b}{a} + \right. \right. \\ \left. \left. + \left( \Lambda_1 + \frac{1}{2} \right) \left( 1 - \frac{a^2}{b^2} \right) \right] \left( 1 + \mu \frac{b}{2\pi R} \right) + \right. \\ \left. + \left[ (2\Lambda_1 + 1) \frac{a^2}{b^2} - 1 \right] \mu \frac{b}{2\pi R} \right\}. \end{aligned} \quad (61)$$

Using Conditions (55a) and (57) analogously, we can refine Expression (49) for the field created by the current in chamber  $J_1$ , with consideration of joints in the housing. The correction, obviously, has the order  $\mu \frac{b}{2\pi R}$ .

5. Pinch in a Poorly Conducting Housing. Having condition opposite to (37), i.e., in the case where the thickness of the skin-layer is larger than the thickness of the housing walls, the asymmetrical (with respect to  $\omega$ ) portion of the magnetic field on the housing must satisfy the condition [10]:

$$H_{\omega}^1 - H_{\omega}^e = \frac{4\pi}{c} \sigma_n d E_{\varphi}, \quad (62)$$

where  $H_{\omega}^e$  is the field outside the housing;  $H_{\omega}^1$  is the field between the housing and the plasma. It is convenient to define these fields in terms of the magnetic current by Formula (23). Considering that

$E_{\varphi} = -\frac{1}{2\pi c r} \frac{d\psi}{dt}$ , from (62) we will obtain the following condition at the housing:

$$\frac{\partial \psi^e}{\partial r} - \frac{\partial \psi^1}{\partial r} = \frac{4\pi d \sigma_n}{c^2} \frac{d\psi}{dt} \quad (63)$$

(with  $\psi$  we designate the value  $\psi^e = \psi^1$  at the housing).

The expressions for  $\psi^i$  and  $\psi^e$  are determined by the formula in (26) (in order to obtain  $\psi^e$ , we take  $a_0 = a_1 = 0$ )

$$\psi^i = -\frac{4\pi R}{c} J \left\{ \ln \frac{8R}{a} - 2 - \frac{a}{2R} \left[ \ln \frac{a}{a} + \left( \Lambda_1 + \frac{1}{2} \right) \left( 1 - \frac{a^2}{a^2} \right) \right] \cos \omega \right\}. \quad (64)$$

$$\psi^e = -\frac{4\pi R}{c} J \left\{ \ln \frac{8R}{a} - 2 + \frac{a}{2R} \left[ \ln \frac{8R}{a} - 1 - \frac{b^2}{a^2} \left( \ln \frac{8R}{a} - 1 \right) - \frac{b^2}{a^2} \left( \Lambda_1 + \frac{1}{2} \right) \left( 1 - \frac{a^2}{b^2} \right) \right] \cos \omega \right\}. \quad (65)$$

If we substitute (64) and (65) into (63) we will obtain the equation for displacement  $\Delta$

$$\begin{aligned} \frac{d}{dt} [J(\Delta - \Delta_0)] &= \\ &= -\frac{c^2 b}{4\pi R \sigma_H d} \left( \ln \frac{8R}{a} + \Lambda_1 - \frac{1}{2} \right) J, \end{aligned} \quad (66)$$

where  $\Delta_0$  is the displacement in an ideally conducting housing (see (48)) where  $B_0 = 0$ :

$$\Delta_0 = \frac{b^2}{2R} \left[ \ln \frac{b}{a} + \left( 1 - \frac{a^2}{b^2} \right) \left( \Lambda_1 + \frac{1}{2} \right) \right]. \quad (67)$$

The characteristic time of holding plasma in the housing, which follows from (66), is determined by the following expression (compare with [11]):

$$t_{re} = \frac{4\pi \sigma_H R d}{c^2} \left( \ln \frac{8R}{a} + \Lambda_1 - \frac{1}{2} \right)^{-1}. \quad (68)$$

In conclusion I wish to thank N. A. Yavlinskiy and his co-workers, especially K. A. Razumova for the interest shown in these calculations.

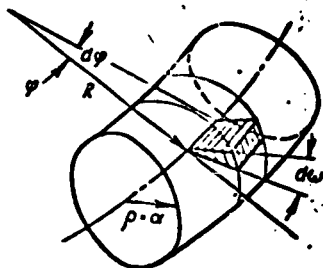


Fig. 1. The volume element, from whose equilibrium condition is determined the distribution of the magnetic field on the pinch surface.

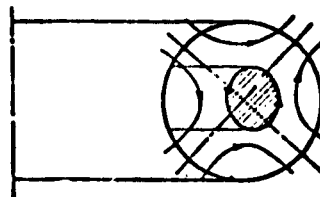


Fig. 2. Configuration of the field causing distortion of the pinch section.

### REFERENCES

1. V. D. Shafranov. "Zh. eksperim. i teor. fiz.", 33, 710, (1957).
2. L. Biermann et al., Z. Naturforsch. 12a, 826 (1957).
3. S. M. Osovets. Fizika plazmy i problema upravlyayemykh termoyadernykh reaktsiy. Vol. 2. M., Izd-vo AN SSSR, 1958, Page 238.
4. V. D. Shafranov. "Zh. eksperim. i teor. fiz." 37, 1088 (1959).
5. Yu. V. Vandakurov. "Zh. Tekhn. Fiz." 29, 1312, (1959).
6. V. D. Shafranov. "Zh. tekhn. fiz." (in print).
7. S. I. Braginskiy, V. D. Shafranov. Fizika plazmy i problema upravlyayemykh termoyadernykh reaktsiy, Vol. 2. M., Izd-vo AN SSSR, page 26, 1958.
8. V. A. Fok. "Zh. ryssk. fiziko-khim. obshch." 62, 218, (1930).
9. M. A. Leontovich. Fizika plazmy i problema upravlyayemykh termoyadernykh reaktsiy. Vol. 1., M., Izd-vo AN SSSR, page 110, 1958.
10. M. A. Leontovich. Fizika plazmy i problema upravlyayemykh termoyadernykh reaktsiy. Vol. 1., M., Izd-vo AN SSSR, page 222, 1958.
11. S. I. Braginskiy, V. D. Shafranov. Tr. Vtoroy mezhdunarodnoy konferentsii po mirnomu ispol'zovaniyu atomnoy energii (Geneva, 1958). Dokl. sovetskikh uchenykh. Vol. 1., M., Atomizdat, page 221, 1959.
12. L. A. Artsimovich, and K. B. Kartashov. "Dokl. AN SSSR," 146, 1305, (1962).
13. V. S. Vacil'evskii and others. "Zh. Tekhn. fiz.", 30, 1135 (1960).

# DISTRIBUTION LIST

DEPARTMENT OF DEFENSE	Nr. Copies	MAJOR AIR COMMANDS	Nr. Copies
		AFSC	1
		SCFDD	25
		DDC	5
		TDBTL	3
HEADQUARTERS USAF		TDBDP	1
		AFMTC (MTW)	1
AFCIN-3D2	1	AFWL (WLF)	2
ARL (ARB)	1	ASD (ASYIM)	1
		ESD (ESY)	1
		RADC (RAY)	2
		SSD (SSF)	
OTHER AGENCIES			
CIA	1		
NSA	6		
DIA	9		
AID	2		
OTS	2		
AEC	2		
PWS	1		
NASA	1		
ARMY (FSTC)	3		
NAVY	3		
NAFEC	1		
RAND	1		
AFCRL (CRCLR)	1		